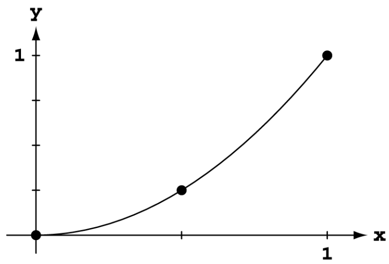
CHAPTER 5 INTEGRATION

5.1 AREA AND ESTIMATING WITH FINITE SUMS

1. $f(x) = x^2$

Since f is increasing on $[0, 1]$, we use left endpoints to obtain lower sums and right endpoints to obtain upper sums.



(a) $\triangle x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\triangle x = \frac{i}{2} \Rightarrow$ a lower sum is $\sum_{i=0}^{1}\left(\frac{i}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2}\left(0^2 + \left(\frac{1}{2}\right)^2\right) = \frac{1}{8}$

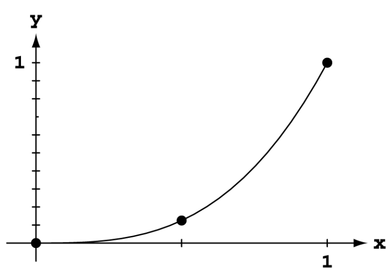
(b) $\triangle x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\triangle x = \frac{i}{4} \Rightarrow$ a lower sum is $\sum_{i=0}^{3}\left(\frac{i}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4}\left(0^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{1}{4} \cdot \frac{7}{8} = \frac{7}{32}$

(c) $\triangle x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\triangle x = \frac{i}{2} \Rightarrow$ an upper sum is $\sum_{i=1}^{2}\left(\frac{i}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + 1^2\right) = \frac{5}{8}$

(d) $\triangle x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\triangle x = \frac{i}{4} \Rightarrow$ an upper sum is $\sum_{i=1}^{4}\left(\frac{i}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right) = \frac{1}{4} \cdot \left(\frac{30}{16}\right) = \frac{15}{32}$

2. $f(x) = x^3$

Since f is increasing on $[0, 1]$, we use left endpoints to obtain lower sums and right endpoints to obtain upper sums.



(a) $\triangle x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\triangle x = \frac{i}{2} \Rightarrow$ a lower sum is $\sum_{i=0}^{1}\left(\frac{i}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{2}\left(0^3 + \left(\frac{1}{2}\right)^3\right) = \frac{1}{16}$

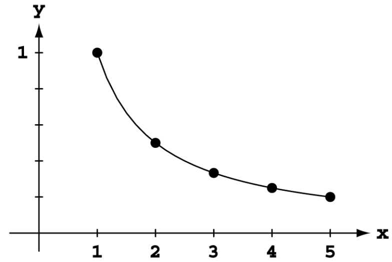
(b) $\triangle x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\triangle x = \frac{i}{4} \Rightarrow$ a lower sum is $\sum_{i=0}^{3}\left(\frac{i}{4}\right)^3 \cdot \frac{1}{4} = \frac{1}{4}\left(0^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3\right) = \frac{36}{256} = \frac{9}{64}$

(c) $\triangle x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\triangle x = \frac{i}{2} \Rightarrow$ an upper sum is $\sum_{i=1}^{2}\left(\frac{i}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^3 + 1^3\right) = \frac{1}{2} \cdot \frac{9}{8} = \frac{9}{16}$

(d) $\triangle x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\triangle x = \frac{i}{4} \Rightarrow$ an upper sum is $\sum_{i=1}^{4}\left(\frac{i}{4}\right)^3 \cdot \frac{1}{4} = \frac{1}{4}\left(\left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3 + 1^3\right) = \ = \frac{100}{256} = \frac{25}{64}$

3. $f(x) = \frac{1}{x}$

Since f is decreasing on $[1, 5]$, we use left endpoints to obtain upper sums and right endpoints to obtain lower sums.



(a) $\triangle x = \frac{5-1}{2} = 2$ and $x_i = 1 + i\triangle x = 1 + 2i \Rightarrow$ a lower sum is $\sum_{i=1}^{2} \frac{1}{x_i} \cdot 2 = 2\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{16}{15}$

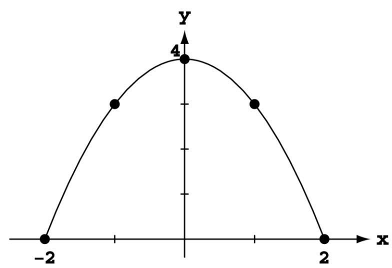
(b) $\triangle x = \frac{5-1}{4} = 1$ and $x_i = 1 + i\triangle x = 1 + i \Rightarrow$ a lower sum is $\sum_{i=1}^{4} \frac{1}{x_i} \cdot 1 = 1\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{77}{60}$

(c) $\triangle x = \frac{5-1}{2} = 2$ and $x_i = 1 + i\triangle x = 1 + 2i \Rightarrow$ an upper sum is $\sum_{i=0}^{1} \frac{1}{x_i} \cdot 2 = 2\left(1 + \frac{1}{3}\right) = \frac{8}{3}$

(d) $\triangle x = \frac{5-1}{4} = 1$ and $x_i = 1 + i\triangle x = 1 + i \Rightarrow$ an upper sum is $\sum_{i=0}^{3} \frac{1}{x_i} \cdot 1 = 1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12}$

4. $f(x) = 4 - x^2$

Since f is increasing on $[-2, 0]$ and decreasing on $[0, 2]$, we use left endpoints on $[-2, 0]$ and right endpoints on $[0, 2]$ to obtain lower sums and use right endpoints on $[-2, 0]$ and left endpoints on $[0, 2]$ to obtain upper sums.



(a) $\triangle x = \frac{2-(-2)}{2} = 2$ and $x_i = -2 + i\triangle x = -2 + 2i \Rightarrow$ a lower sum is $2 \cdot \left(4 - (-2)^2\right) + 2 \cdot (4 - 2^2) = 0$

(b) $\triangle x = \frac{2-(-2)}{4} = 1$ and $x_i = -2 + i\triangle x = -2 + i \Rightarrow$ a lower sum is $\sum_{i=0}^{1} \left(4 - (x_i)^2\right) \cdot 1 + \sum_{i=3}^{4} \left(4 - (x_i)^2\right) \cdot 1$

$= 1\left(\left(4 - (-2)^2\right) + \left(4 - (-1)^2\right) + (4 - 1^2) + (4 - 2^2)\right) = 6$

(c) $\triangle x = \frac{2-(-2)}{2} = 2$ and $x_i = -2 + i\triangle x = -2 + 2i \Rightarrow$ a upper sum is $2 \cdot \left(4 - (0)^2\right) + 2 \cdot (4 - 0^2) = 16$

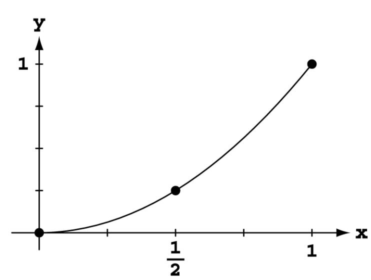
(d) $\triangle x = \frac{2-(-2)}{4} = 1$ and $x_i = -2 + i\triangle x = -2 + i \Rightarrow$ a upper sum is $\sum_{i=1}^{2} \left(4 - (x_i)^2\right) \cdot 1 + \sum_{i=2}^{3} \left(4 - (x_i)^2\right) \cdot 1$

$= 1\left(\left(4 - (-1)^2\right) + (4 - 0^2) + (4 - 0^2) + (4 - 1^2)\right) = 14$

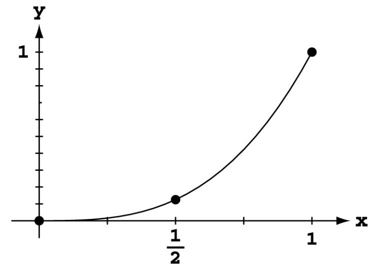
5. $f(x) = x^2$

Using 2 rectangles $\Rightarrow \triangle x = \frac{1-0}{2} = \frac{1}{2} \Rightarrow \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right)$
$= \frac{1}{2}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{10}{32} = \frac{5}{16}$

Using 4 rectangles $\Rightarrow \triangle x = \frac{1-0}{4} = \frac{1}{4}$
$\Rightarrow \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$
$= \frac{1}{4}\left(\left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{5}{8}\right)^2 + \left(\frac{7}{8}\right)^2\right) = \frac{21}{64}$



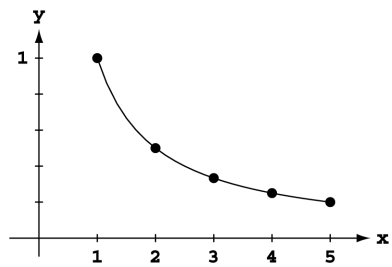
6. $f(x) = x^3$



Using 2 rectangles $\Rightarrow \triangle x = \frac{1-0}{2} = \frac{1}{2} \Rightarrow \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right)$
$= \frac{1}{2}\left(\left(\frac{1}{4}\right)^3 + \left(\frac{3}{4}\right)^3\right) = \frac{28}{2\cdot 64} = \frac{7}{32}$

Using 4 rectangles $\Rightarrow \triangle x = \frac{1-0}{4} = \frac{1}{4}$
$\Rightarrow \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$
$= \frac{1}{4}\left(\frac{1^3+3^3+5^3+7^3}{8^3}\right) = \frac{496}{4\cdot 8^3} = \frac{124}{8^3} = \frac{31}{128}$

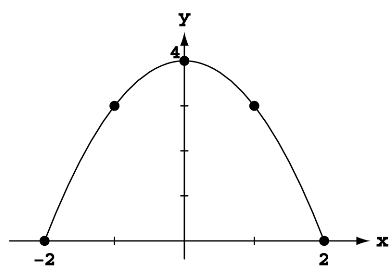
7. $f(x) = \frac{1}{x}$



Using 2 rectangles $\Rightarrow \triangle x = \frac{5-1}{2} = 2 \Rightarrow 2(f(2) + f(4))$
$= 2\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{2}$

Using 4 rectangles $\Rightarrow \triangle x = \frac{5-1}{4} = 1$
$\Rightarrow 1\left(f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{9}{2}\right)\right)$
$= 1\left(\frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9}\right) = \frac{1488}{3\cdot 5\cdot 7\cdot 9} = \frac{496}{5\cdot 7\cdot 9} = \frac{496}{315}$

8. $f(x) = 4 - x^2$



Using 2 rectangles $\Rightarrow \triangle x = \frac{2-(-2)}{2} = 2 \Rightarrow 2(f(-1) + f(1))$
$= 2(3+3) = 12$

Using 4 rectangles $\Rightarrow \triangle x = \frac{2-(-2)}{4} = 1$
$\Rightarrow 1\left(f\left(-\frac{3}{2}\right) + f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\right)$
$= 1\left(\left(4 - \left(-\frac{3}{2}\right)^2\right) + \left(4 - \left(-\frac{1}{2}\right)^2\right) + \left(4 - \left(\frac{1}{2}\right)^2\right) + \left(4 - \left(\frac{3}{2}\right)^2\right)\right)$
$= 16 - \left(\frac{9}{4}\cdot 2 + \frac{1}{4}\cdot 2\right) = 16 - \frac{10}{2} = 11$

9. (a) $D \approx (0)(1) + (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) = 87$ inches
(b) $D \approx (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) + (0)(1) = 87$ inches

10. (a) $D \approx (1)(300) + (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) = 5220$ meters (NOTE: 5 minutes = 300 seconds)
(b) $D \approx (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) + (0)(300) = 4920$ meters (NOTE: 5 minutes = 300 seconds)

11. (a) $D \approx (0)(10) + (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10) + (35)(10) + (44)(10) + (30)(10) = 3490$ feet ≈ 0.66 miles
(b) $D \approx (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10) + (35)(10) + (44)(10) + (30)(10) + (35)(10) = 3840$ feet ≈ 0.73 miles

12. (a) The distance traveled will be the area under the curve. We will use the approximate velocities at the midpoints of each time interval to approximate this area using rectangles. Thus,
$D \approx (20)(0.001) + (50)(0.001) + (72)(0.001) + (90)(0.001) + (102)(0.001) + (112)(0.001) + (120)(0.001) + (128)(0.001) + (134)(0.001) + (139)(0.001) \approx 0.967$ miles
(b) Roughly, after 0.0063 hours, the car would have gone 0.484 miles, where 0.0060 hours = 22.7 sec. At 22.7 sec, the velocity was approximately 120 mi/hr.

13. (a) Because the acceleration is decreasing, an upper estimate is obtained using left end-points in summing acceleration $\cdot\ \Delta t$. Thus, $\Delta t = 1$ and speed $\approx [32.00 + 19.41 + 11.77 + 7.14 + 4.33](1) = 74.65$ ft/sec

(b) Using right end-points we obtain a lower estimate: speed $\approx [19.41 + 11.77 + 7.14 + 4.33 + 2.63](1) = 45.28$ ft/sec

(c) Upper estimates for the speed at each second are:

t	0	1	2	3	4	5
v	0	32.00	51.41	63.18	70.32	74.65

Thus, the distance fallen when $t = 3$ seconds is $s \approx [32.00 + 51.41 + 63.18](1) = 146.59$ ft.

14. (a) The speed is a decreasing function of time $\Rightarrow$ right end-points give an lower estimate for the height (distance) attained. Also

t	0	1	2	3	4	5
v	400	368	336	304	272	240

gives the time-velocity table by subtracting the constant $g = 32$ from the speed at each time increment $\Delta t = 1$ sec. Thus, the speed ≈ 240 ft/sec after 5 seconds.

(b) A lower estimate for height attained is $h \approx [368 + 336 + 304 + 272 + 240](1) = 1520$ ft.

15. Partition $[0, 2]$ into the four subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are $f(m_1) = (0.25)^3 = \frac{1}{64}$, $f(m_2) = (0.75)^3 = \frac{27}{64}$, $f(m_3) = (1.25)^3 = \frac{125}{64}$, and $f(m_4) = (1.75)^3 = \frac{343}{64}$

Notice that the average value is approximated by $\frac{1}{2}\left[\left(\frac{1}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{7}{4}\right)^3\left(\frac{1}{2}\right)\right] = \frac{31}{16}$

$= \frac{1}{\text{length of }[0,2]} \cdot \left[\begin{array}{c}\text{approximate area under}\\ \text{curve } f(x) = x^3\end{array}\right]$. We use this observation in solving the next several exercises.

16. Partition $[1, 9]$ into the four subintervals $[1, 3]$, $[3, 5]$, $[5, 7]$, and $[7, 9]$. The midpoints of these subintervals are $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, and $m_4 = 8$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2}$, $f(m_2) = \frac{1}{4}$, $f(m_3) = \frac{1}{6}$, and $f(m_4) = \frac{1}{8}$. The width of each rectangle is $\Delta x = 2$. Thus,
Area $\approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{6}\right) + 2\left(\frac{1}{8}\right) = \frac{25}{12} \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of }[1,9]} = \frac{\left(\frac{25}{12}\right)}{8} = \frac{25}{96}$.

17. Partition $[0, 2]$ into the four subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of the subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are
$f(m_1) = \frac{1}{2} + \sin^2\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$, $f(m_2) = \frac{1}{2} + \sin^2\frac{3\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$, $f(m_3) = \frac{1}{2} + \sin^2\frac{5\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2$
$= \frac{1}{2} + \frac{1}{2} = 1$, and $f(m_4) = \frac{1}{2} + \sin^2\frac{7\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$. The width of each rectangle is $\Delta x = \frac{1}{2}$. Thus,
Area $\approx (1 + 1 + 1 + 1)\left(\frac{1}{2}\right) = 2 \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of }[0,2]} = \frac{2}{2} = 1$.

18. Partition $[0, 4]$ into the four subintervals $[0, 1]$, $[1, 2,]$, $[2, 3]$, and $[3, 4]$. The midpoints of the subintervals are $m_1 = \frac{1}{2}$, $m_2 = \frac{3}{2}$, $m_3 = \frac{5}{2}$, and $m_4 = \frac{7}{2}$. The heights of the four approximating rectangles are
$f(m_1) = 1 - \left(\cos\left(\frac{\pi\left(\frac{1}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{\pi}{8}\right)\right)^4 = 0.27145$ (to 5 decimal places),
$f(m_2) = 1 - \left(\cos\left(\frac{\pi\left(\frac{3}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{3\pi}{8}\right)\right)^4 = 0.97855$, $f(m_3) = 1 - \left(\cos\left(\frac{\pi\left(\frac{5}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{5\pi}{8}\right)\right)^4$
$= 0.97855$, and $f(m_4) = 1 - \left(\cos\left(\frac{\pi\left(\frac{7}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{7\pi}{8}\right)\right)^4 = 0.27145$. The width of each rectangle is $\Delta x = 1$. Thus, Area $\approx (0.27145)(1) + (0.97855)(1) + (0.97855)(1) + (0.27145)(1) = 2.5 \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of }[0,4]} = \frac{2.5}{4} = \frac{5}{8}$.

19. Since the leakage is increasing, an upper estimate uses right endpoints and a lower estimate uses left endpoints:
   (a) upper estimate $= (70)(1) + (97)(1) + (136)(1) + (190)(1) + (265)(1) = 758$ gal,
   lower estimate $= (50)(1) + (70)(1) + (97)(1) + (136)(1) + (190)(1) = 543$ gal.
   (b) upper estimate $= (70 + 97 + 136 + 190 + 265 + 369 + 516 + 720) = 2363$ gal,
   lower estimate $= (50 + 70 + 97 + 136 + 190 + 265 + 369 + 516) = 1693$ gal.
   (c) worst case: $2363 + 720t = 25{,}000 \Rightarrow t \approx 31.4$ hrs;
   best case: $1693 + 720t = 25{,}000 \Rightarrow t \approx 32.4$ hrs

20. Since the pollutant release increases over time, an upper estimate uses right endpoints and a lower estimate uses left endpoints:
   (a) upper estimate $= (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) + (0.52)(30) = 60.9$ tons
   lower estimate $= (0.05)(30) + (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) = 46.8$ tons
   (b) Using the lower (best case) estimate: $46.8 + (0.52)(30) + (0.63)(30) + (0.70)(30) + (0.81)(30) = 126.6$ tons, so near the end of September 125 tons of pollutants will have been released.

21. (a) The diagonal of the square has length 2, so the side length is $\sqrt{2}$. Area $= \left(\sqrt{2}\right)^2 = 2$
   (b) Think of the octagon as a collection of 16 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{16} = \frac{\pi}{8}$.
   Area $= 16\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = 4\sin\frac{\pi}{4} = 2\sqrt{2} \approx 2.828$
   (c) Think of the 16-gon as a collection of 32 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{32} = \frac{\pi}{16}$.
   Area $= 32\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{16}\right)\left(\cos\frac{\pi}{16}\right) = 8\sin\frac{\pi}{8} = 2\sqrt{2} \approx 3.061$
   (d) Each area is less than the area of the circle, π. As n increases, the area approaches π.

22. (a) Each of the isosceles triangles is made up of two right triangles having hypotenuse 1 and an acute angle measuring $\frac{2\pi}{2n} = \frac{\pi}{n}$. The area of each isosceles triangle is $A_T = 2\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{n}\right)\left(\cos\frac{\pi}{n}\right) = \frac{1}{2}\sin\frac{2\pi}{n}$.
   (b) The area of the polygon is $A_P = nA_T = \frac{n}{2}\sin\frac{2\pi}{n}$, so $\lim\limits_{n\to\infty}\frac{n}{2}\sin\frac{2\pi}{n} = \lim\limits_{n\to\infty}\pi\cdot\frac{\sin\frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)} = \pi$
   (c) Multiply each area by r^2.
   $A_T = \frac{1}{2}r^2\sin\frac{2\pi}{n}$
   $A_P = \frac{n}{2}r^2\sin\frac{2\pi}{n}$
   $\lim\limits_{n\to\infty} A_P = \pi r^2$

23-26. Example CAS commands:
<u>Maple</u>:

```
with( Student[Calculus1] );
f := x -> sin(x);
a := 0;
b := Pi;
plot( f(x), x=a..b, title="#23(a) (Section 5.1)" );
N := [ 100, 200, 1000 ];                                  # (b)
for n in N do
  Xlist := [ a+1.*(b-a)/n*i $ i=0..n ];
  Ylist := map( f, Xlist );
end do:
for n in N do                                             # (c)
```

```
  Avg[n] := evalf(add(y,y=Ylist)/nops(Ylist));
end do;
avg := FunctionAverage( f(x), x=a..b, output=value );
evalf( avg );
FunctionAverage(f(x),x=a..b,output=plot);      # (d)
fsolve( f(x)=avg, x=0.5 );
fsolve( f(x)=avg, x=2.5 );
fsolve( f(x)=Avg[1000], x=0.5 );
fsolve( f(x)=Avg[1000], x=2.5 );
```

Mathematica: (assigned function and values for a and b may vary):

Symbols for π, $\rightarrow$, powers, roots, fractions, etc. are available in Palettes (under File).

Never insert a space between the name of a function and its argument.

```
Clear[x]
f[x_]:=x  Sin[1/x]
{a,b}={π/4, π}
Plot[f[x],{x, a, b}]
```

The following code computes the value of the function for each interval midpoint and then finds the average. Each sequence of commands for a different value of n (number of subdivisions) should be placed in a separate cell.

```
n =100; dx = (b − a) /n;
values = Table[N[f[x]], {x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
n =200; dx = (b − a) /n;
values = Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
n =1000; dx = (b − a) /n;
values = Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
FindRoot[f[x] == average,{x, a}]
```

5.2 SIGMA NOTATION AND LIMITS OF FINITE SUMS

1. $\sum_{k=1}^{2} \frac{6k}{k+1} = \frac{6(1)}{1+1} + \frac{6(2)}{2+1} = \frac{6}{2} + \frac{12}{3} = 7$

2. $\sum_{k=1}^{3} \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

3. $\sum_{k=1}^{4} \cos k\pi = \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1 + 1 - 1 + 1 = 0$

4. $\sum_{k=1}^{5} \sin k\pi = \sin(1\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) = 0 + 0 + 0 + 0 + 0 = 0$

5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k} = (-1)^{1+1} \sin \frac{\pi}{1} + (-1)^{2+1} \sin \frac{\pi}{2} + (-1)^{3+1} \sin \frac{\pi}{3} = 0 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-2}{2}$

6. $\sum_{k=1}^{4} (-1)^{k} \cos k\pi = (-1)^{1} \cos(1\pi) + (-1)^{2} \cos(2\pi) + (-1)^{3} \cos(3\pi) + (-1)^{4} \cos(4\pi)$

   $= -(-1) + 1 - (-1) + 1 = 4$

7. (a) $\sum_{k=1}^{6} 2^{k-1} = 2^{1-1} + 2^{2-1} + 2^{3-1} + 2^{4-1} + 2^{5-1} + 2^{6-1} = 1 + 2 + 4 + 8 + 16 + 32$

(b) $\sum_{k=0}^{5} 2^{k} = 2^{0} + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} = 1 + 2 + 4 + 8 + 16 + 32$

(c) $\sum_{k=-1}^{4} 2^{k+1} = 2^{-1+1} + 2^{0+1} + 2^{1+1} + 2^{2+1} + 2^{3+1} + 2^{4+1} = 1 + 2 + 4 + 8 + 16 + 32$

All of them represent $1 + 2 + 4 + 8 + 16 + 32$

8. (a) $\sum_{k=1}^{6} (-2)^{k-1} = (-2)^{1-1} + (-2)^{2-1} + (-2)^{3-1} + (-2)^{4-1} + (-2)^{5-1} + (-2)^{6-1} = 1 - 2 + 4 - 8 + 16 - 32$

(b) $\sum_{k=0}^{5} (-1)^{k}2^{k} = (-1)^{0}2^{0} + (-1)^{1}2^{1} + (-1)^{2}2^{2} + (-1)^{3}2^{3} + (-1)^{4}2^{4} + (-1)^{5}2^{5} = 1 - 2 + 4 - 8 + 16 - 32$

(c) $\sum_{k=-2}^{3} (-1)^{k+1}2^{k+2} = (-1)^{-2+1}2^{-2+2} + (-1)^{-1+1}2^{-1+2} + (-1)^{0+1}2^{0+2} + (-1)^{1+1}2^{1+2} + (-1)^{2+1}2^{2+2}$

$+ (-1)^{3+1}2^{3+2} = -1 + 2 - 4 + 8 - 16 + 32;$

(a) and (b) represent $1 - 2 + 4 - 8 + 16 - 32$; (c) is not equivalent to the other two

9. (a) $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1} = \frac{(-1)^{2-1}}{2-1} + \frac{(-1)^{3-1}}{3-1} + \frac{(-1)^{4-1}}{4-1} = -1 + \frac{1}{2} - \frac{1}{3}$

(b) $\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1} = \frac{(-1)^{0}}{0+1} + \frac{(-1)^{1}}{1+1} + \frac{(-1)^{2}}{2+1} = 1 - \frac{1}{2} + \frac{1}{3}$

(c) $\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2} = \frac{(-1)^{-1}}{-1+2} + \frac{(-1)^{0}}{0+2} + \frac{(-1)^{1}}{1+2} = -1 + \frac{1}{2} - \frac{1}{3}$

(a) and (c) are equivalent; (b) is not equivalent to the other two.

10. (a) $\sum_{k=1}^{4} (k-1)^{2} = (1-1)^{2} + (2-1)^{2} + (3-1)^{2} + (4-1)^{2} = 0 + 1 + 4 + 9$

(b) $\sum_{k=-1}^{3} (k+1)^{2} = (-1+1)^{2} + (0+1)^{2} + (1+1)^{2} + (2+1)^{2} + (3+1)^{2} = 0 + 1 + 4 + 9 + 16$

(c) $\sum_{k=-3}^{-1} k^{2} = (-3)^{2} + (-2)^{2} + (-1)^{2} = 9 + 4 + 1$

(a) and (c) are equivalent to each other; (b) is not equivalent to the other two.

11. $\sum_{k=1}^{6} k$

12. $\sum_{k=1}^{4} k^{2}$

13. $\sum_{k=1}^{4} \frac{1}{2^{k}}$

14. $\sum_{k=1}^{5} 2k$

15. $\sum_{k=1}^{5} (-1)^{k+1} \frac{1}{k}$

16. $\sum_{k=1}^{5} (-1)^{k} \frac{k}{5}$

17. (a) $\sum_{k=1}^{n} 3a_k = 3\sum_{k=1}^{n} a_k = 3(-5) = -15$

(b) $\sum_{k=1}^{n} \frac{b_k}{6} = \frac{1}{6}\sum_{k=1}^{n} b_k = \frac{1}{6}(6) = 1$

(c) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = -5 + 6 = 1$

(d) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = -5 - 6 = -11$

(e) $\sum_{k=1}^{n} (b_k - 2a_k) = \sum_{k=1}^{n} b_k - 2\sum_{k=1}^{n} a_k = 6 - 2(-5) = 16$

18. (a) $\sum_{k=1}^{n} 8a_k = 8\sum_{k=1}^{n} a_k = 8(0) = 0$ (b) $\sum_{k=1}^{n} 250b_k = 250\sum_{k=1}^{n} b_k = 250(1) = 250$

(c) $\sum_{k=1}^{n} (a_k + 1) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} 1 = 0 + n = n$ (d) $\sum_{k=1}^{n} (b_k - 1) = \sum_{k=1}^{n} b_k - \sum_{k=1}^{n} 1 = 1 - n$

19. (a) $\sum_{k=1}^{10} k = \frac{10(10+1)}{2} = 55$ (b) $\sum_{k=1}^{10} k^2 = \frac{10(10+1)(2(10)+1)}{6} = 385$

(c) $\sum_{k=1}^{10} k^3 = \left[\frac{10(10+1)}{2}\right]^2 = 55^2 = 3025$

20. (a) $\sum_{k=1}^{13} k = \frac{13(13+1)}{2} = 91$ (b) $\sum_{k=1}^{13} k^2 = \frac{13(13+1)(2(13)+1)}{6} = 819$

(c) $\sum_{k=1}^{13} k^3 = \left[\frac{13(13+1)}{2}\right]^2 = 91^2 = 8281$

21. $\sum_{k=1}^{7} -2k = -2\sum_{k=1}^{7} k = -2\left(\frac{7(7+1)}{2}\right) = -56$

22. $\sum_{k=1}^{5} \frac{\pi k}{15} = \frac{\pi}{15}\sum_{k=1}^{5} k = \frac{\pi}{15}\left(\frac{5(5+1)}{2}\right) = \pi$

23. $\sum_{k=1}^{6} (3 - k^2) = \sum_{k=1}^{6} 3 - \sum_{k=1}^{6} k^2 = 3(6) - \frac{6(6+1)(2(6)+1)}{6} = -73$

24. $\sum_{k=1}^{6} (k^2 - 5) = \sum_{k=1}^{6} k^2 - \sum_{k=1}^{6} 5 = \frac{6(6+1)(2(6)+1)}{6} - 5(6) = 61$

25. $\sum_{k=1}^{5} k(3k+5) = \sum_{k=1}^{5} (3k^2 + 5k) = 3\sum_{k=1}^{5} k^2 + 5\sum_{k=1}^{5} k = 3\left(\frac{5(5+1)(2(5)+1)}{6}\right) + 5\left(\frac{5(5+1)}{2}\right) = 240$

26. $\sum_{k=1}^{7} k(2k+1) = \sum_{k=1}^{7} (2k^2 + k) = 2\sum_{k=1}^{7} k^2 + \sum_{k=1}^{7} k = 2\left(\frac{7(7+1)(2(7)+1)}{6}\right) + \frac{7(7+1)}{2} = 308$

27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3 = \frac{1}{225}\sum_{k=1}^{5} k^3 + \left(\sum_{k=1}^{5} k\right)^3 = \frac{1}{225}\left(\frac{5(5+1)}{2}\right)^2 + \left(\frac{5(5+1)}{2}\right)^3 = 3376$

28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4} = \left(\sum_{k=1}^{7} k\right)^2 - \frac{1}{4}\sum_{k=1}^{7} k^3 = \left(\frac{7(7+1)}{2}\right)^2 - \frac{1}{4}\left(\frac{7(7+1)}{2}\right)^2 = 588$

29. (a) $\sum_{k=1}^{7} 3 = 3(7) = 21$ (b) $\sum_{k=1}^{500} 7 = 7(500) = 3500$

(c) Let $j = k - 2 \Rightarrow k = j + 2$; if $k = 3 \Rightarrow j = 1$ and if $k = 264 \Rightarrow j = 262 \Rightarrow \sum_{k=3}^{264} 10 = \sum_{j=1}^{262} 10 = 10(262) = 2620$

30. (a) Let $j = k - 8 \Rightarrow k = j + 8$; if $k = 9 \Rightarrow j = 1$ and if $k = 36 \Rightarrow j = 28 \Rightarrow \sum_{k=9}^{36} k = \sum_{j=1}^{28} (j+8) = \sum_{j=1}^{28} j + \sum_{j=1}^{28} 8$

$= \frac{28(28+1)}{2} + 8(28) = 630$

(b) Let $j = k - 2 \Rightarrow k = j + 2$; if $k = 3 \Rightarrow j = 1$ and if $k = 17 \Rightarrow j = 15 \Rightarrow \sum_{k=3}^{17} k^2 = \sum_{j=1}^{15} (j+2)^2$

$= \sum_{j=1}^{15} (j^2 + 4j + 4) = \sum_{j=1}^{15} j^2 + \sum_{j=1}^{15} 4j + \sum_{j=1}^{15} 4 = \frac{15(15+1)(2(15)+1)}{6} + 4 \cdot \frac{15(15+1)}{2} + 4(15)$

$= 1240 + 480 + 60 = 1780$

(c) Let $j = k - 17 \Rightarrow k = j + 17$; if $k = 18 \Rightarrow j = 1$ and if $k = 71 \Rightarrow j = 54 \Rightarrow \sum_{k=3}^{71} k(k-1)$

$$= \sum_{j=1}^{54} (j+17)((j+17)-1) = \sum_{j=1}^{54} (j^2 + 33j + 272) = \sum_{j=1}^{54} j^2 + \sum_{j=1}^{54} 33j + \sum_{j=1}^{54} 272$$

$$= \frac{54(54+1)(2(54)+1)}{6} + 33 \cdot \frac{54(54+1)}{2} + 272(54) = 53955 + 49005 + 14688 = 117648$$

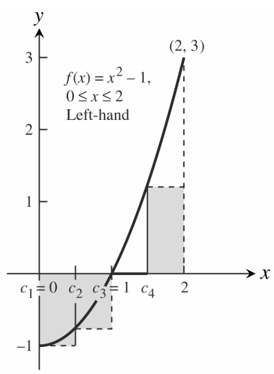
31. (a) $\sum_{k=1}^{n} 4 = 4n$ (b) $\sum_{k=1}^{n} c = cn$

(c) $\sum_{k=1}^{n} (k-1) = \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = \frac{n(n+1)}{2} - n = \frac{n^2 - n}{2}$

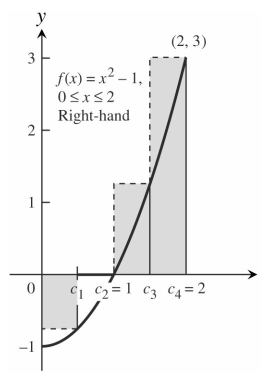
32. (a) $\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right) = \left(\frac{1}{n} + 2n\right)n = 1 + 2n^2$ (b) $\sum_{k=1}^{n} \frac{c}{n} = \frac{c}{n} \cdot n = c$

(c) $\sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n}$

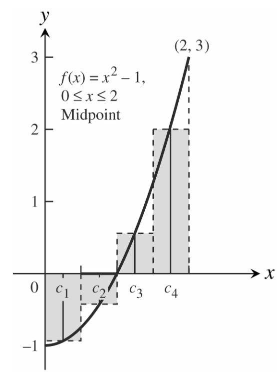
33. (a)



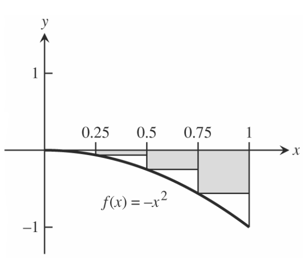
(b)



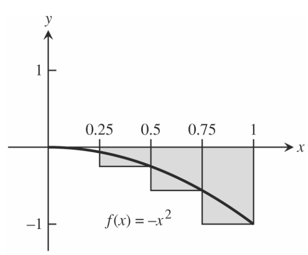
(c)



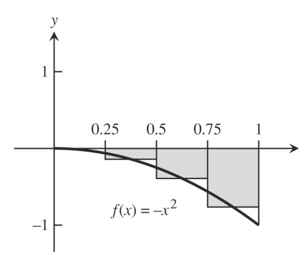
34. (a)



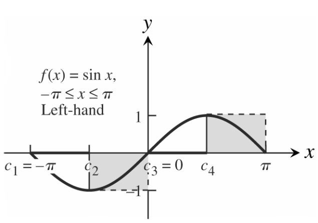
(b)



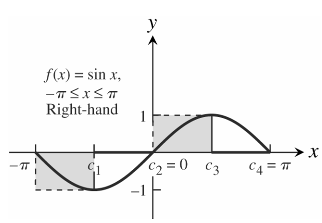
(c)



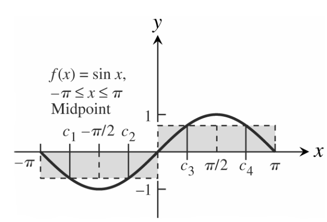
35. (a)



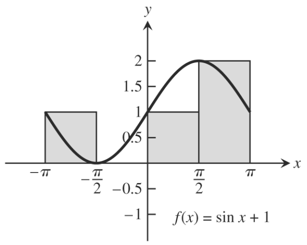
(b)



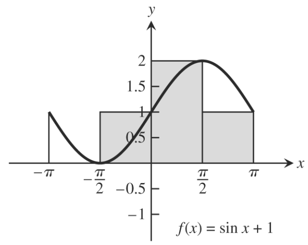
(c)



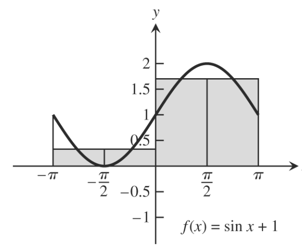
36. (a)



(b)



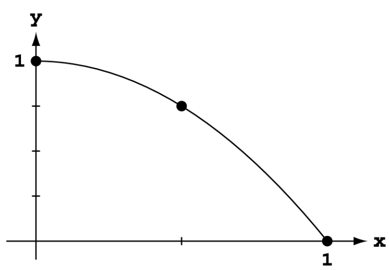
(c)



37. $|x_1 - x_0| = |1.2 - 0| = 1.2$, $|x_2 - x_1| = |1.5 - 1.2| = 0.3$, $|x_3 - x_2| = |2.3 - 1.5| = 0.8$, $|x_4 - x_3| = |2.6 - 2.3| = 0.3$, and $|x_5 - x_4| = |3 - 2.6| = 0.4$; the largest is $\|P\| = 1.2$.

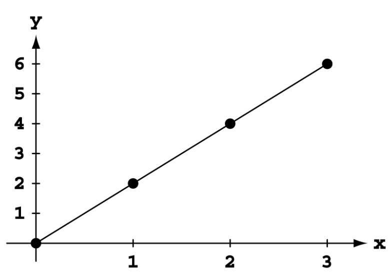
38. $|x_1 - x_0| = |-1.6 - (-2)| = 0.4$, $|x_2 - x_1| = |-0.5 - (-1.6)| = 1.1$, $|x_3 - x_2| = |0 - (-0.5)| = 0.5$, $|x_4 - x_3| = |0.8 - 0| = 0.8$, and $|x_5 - x_4| = |1 - 0.8| = 0.2$; the largest is $\|P\| = 1.1$.

39. $f(x) = 1 - x^2$



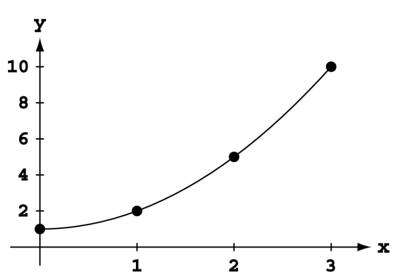
Let $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\triangle x = \frac{i}{n}$. The right-hand sum is
$$\sum_{i=1}^{n}(1 - c_i^2)\tfrac{1}{n} = \tfrac{1}{n}\sum_{i=1}^{n}\left(1 - \left(\tfrac{i}{n}\right)^2\right) = \tfrac{1}{n^3}\sum_{i=1}^{n}(n^2 - i^2)$$
$$= \tfrac{n^3}{n^3} - \tfrac{1}{n^3}\sum_{i=1}^{n} i^2 = 1 - \tfrac{n(n+1)(2n+1)}{6n^3} = 1 - \tfrac{2n^3+3n^2+n}{6n^3}$$
$$= 1 - \tfrac{2+\frac{3}{n}+\frac{1}{n^2}}{6}.\ \text{Thus,}\quad \lim_{n\to\infty}\sum_{i=1}^{n}(1 - c_i^2)\tfrac{1}{n}$$
$$= \lim_{n\to\infty}\left(1 - \tfrac{2-\frac{3}{n}+\frac{1}{n^2}}{6}\right) = 1 - \tfrac{1}{3} = \tfrac{2}{3}$$

40. $f(x) = 2x$



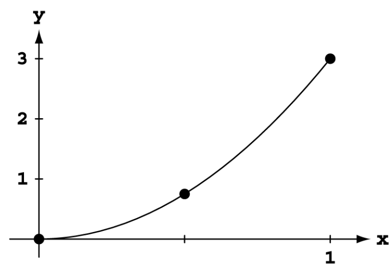
Let $\triangle x = \frac{3-0}{n} = \frac{3}{n}$ and $c_i = i\triangle x = \frac{3i}{n}$. The right-hand sum is
$$\sum_{i=1}^{n} 2c_i\left(\tfrac{3}{n}\right) = \sum_{i=1}^{n}\tfrac{6i}{n}\cdot\tfrac{3}{n} = \tfrac{18}{n^2}\sum_{i=1}^{n} i = \tfrac{18}{n^2}\cdot\tfrac{n(n+1)}{2} = \tfrac{9n^2+9n}{n^2}$$
Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6i}{n}\cdot\frac{3}{n} = \lim_{n\to\infty}\frac{9n^2+9n}{n^2} = \lim_{n\to\infty}\left(9+\frac{9}{n}\right) = 9.$

41. $f(x) = x^2 + 1$



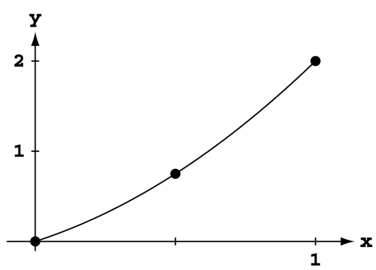
Let $\triangle x = \frac{3-0}{n} = \frac{3}{n}$ and $c_i = i\triangle x = \frac{3i}{n}$. The right-hand sum is
$$\sum_{i=1}^{n}(c_i^2+1)\tfrac{3}{n} = \sum_{i=1}^{n}\left(\left(\tfrac{3i}{n}\right)^2+1\right)\tfrac{3}{n} = \tfrac{3}{n}\sum_{i=1}^{n}\left(\tfrac{9i^2}{n^2}+1\right)$$
$$= \tfrac{27}{n}\sum_{i=1}^{n} i^2 + \tfrac{3}{n}\cdot n = \tfrac{27}{n^3}\left(\tfrac{n(n+1)(2n+1)}{6}\right) + 3$$
$$= \tfrac{9(2n^3+3n^2+n)}{2n^3} + 3 = \tfrac{18+\frac{27}{n}+\frac{9}{n^2}}{2} + 3.\ \text{Thus,}$$
$$\lim_{n\to\infty}\sum_{i=1}^{n}(c_i^2+1)\tfrac{3}{n} = \lim_{n\to\infty}\left(\tfrac{18+\frac{27}{n}+\frac{9}{n^2}}{2}+3\right) = 9+3 = 12.$$

42. $f(x) = 3x^2$



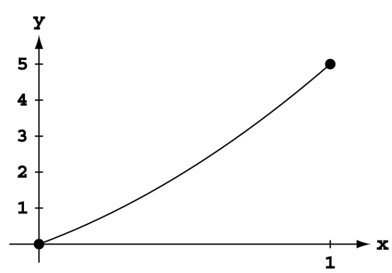
Let $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\triangle x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n} 3c_i^2\left(\frac{1}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \frac{3}{n^3}\sum_{i=1}^{n} i^2 = \frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$
$= \frac{2n^3+3n^2+n}{2n^3} = \frac{2+\frac{3}{n}+\frac{1}{n^2}}{2}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n} 3c_i^2\left(\frac{1}{n}\right)$
$= \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{2}\right) = \frac{2}{2} = 1.$

43. $f(x) = x + x^2 = x(1+x)$



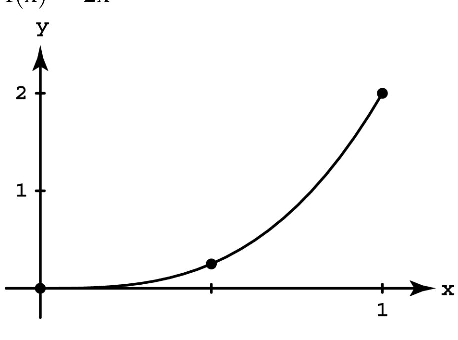
Let $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\triangle x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n}\left(c_i + c_i^2\right)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{i}{n} + \left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2$
$= \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{n^2+n}{2n^2} + \frac{2n^3+3n^2+n}{6n^3}$
$= \frac{1+\frac{1}{n}}{2} + \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(c_i + c_i^2\right)\frac{1}{n}$
$= \lim_{n\to\infty}\left[\left(\frac{1+\frac{1}{n}}{2}\right) + \left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right)\right] = \frac{1}{2} + \frac{2}{6} = \frac{5}{6}.$

44. $f(x) = 3x + 2x^2$



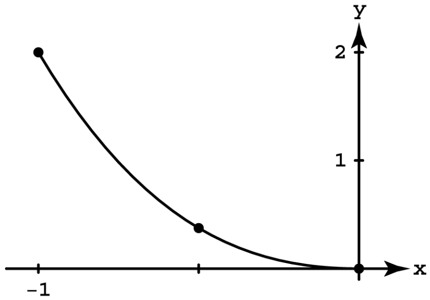
Let $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\triangle x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n}\left(3c_i + 2c_i^2\right)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{3i}{n} + 2\left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \frac{3}{n^2}\sum_{i=1}^{n} i + \frac{2}{n^3}\sum_{i=1}^{n} i^2$
$= \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{3n^2+3n}{2n^2} + \frac{2n^2+3n+1}{3n^2}$
$= \frac{3+\frac{3}{n}}{2} + \frac{2+\frac{3}{n}+\frac{1}{n^2}}{3}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(3c_i + 2c_i^2\right)\frac{1}{n}$
$= \lim_{n\to\infty}\left[\left(\frac{3+\frac{3}{n}}{2}\right) + \left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{3}\right)\right] = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}.$

45. $f(x) = 2x^3$



Let $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\triangle x = \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n}\left(2c_i^3\right)\frac{1}{n} = \sum_{i=1}^{n}\left(2\left(\frac{i}{n}\right)^3\right)\frac{1}{n} = \frac{2}{n^4}\sum_{i=1}^{n} i^3 = \frac{2}{n^4}\left(\frac{n(n+1)}{2}\right)^2$
$= \frac{2n^2(n^2+2n+1)}{4n^4} = \frac{n^2+2n+1}{2n^2} = \frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}$.
Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2c_i^3\right)\frac{1}{n} = \lim_{n\to\infty}\left[\frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}\right] = \frac{1}{2}.$

46. $f(x) = x^2 - x^3$



Let $\triangle x = \frac{0-(-1)}{n} = \frac{1}{n}$ and $c_i = -1 + i\triangle x = -1 + \frac{i}{n}$. The right-hand sum is $\sum_{i=1}^{n}(c_i^2 - c_i^3)\frac{1}{n} = \sum_{i=1}^{n}\left(\left(-1+\frac{i}{n}\right)^2 - \left(-1+\frac{i}{n}\right)^3\right)\frac{1}{n}$

$= \sum_{i=1}^{n}\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{2}{n} - \frac{5i}{n^2} + \frac{4i^2}{n^3} - \frac{i^3}{n^4}\right)$

$= \sum_{i=1}^{n}\frac{2}{n} - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$

$= \frac{2}{n}(n) - \frac{5}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{4}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$

$= 2 - \frac{5n+5}{2n} + \frac{4n^2+6n+2}{3n^2} - \frac{n^2+2n+1}{4n^2} = 2 - \frac{5+\frac{5}{n}}{2} + \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3} - \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}$. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}(c_i^2 - c_i^3)\frac{1}{n}$

$= \lim_{n\to\infty}\left[2 - \frac{5+\frac{5}{n}}{2} + \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3} - \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}\right] = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$.

5.3 THE DEFINITE INTEGRAL

1. $\int_0^2 x^2\,dx$

2. $\int_{-1}^0 2x^3\,dx$

3. $\int_{-7}^5 (x^2 - 3x)\,dx$

4. $\int_1^4 \frac{1}{x}\,dx$

5. $\int_2^3 \frac{1}{1-x}\,dx$

6. $\int_0^1 \sqrt{4-x^2}\,dx$

7. $\int_{-\pi/4}^0 (\sec x)\,dx$

8. $\int_0^{\pi/4} (\tan x)\,dx$

9. (a) $\int_2^2 g(x)\,dx = 0$

(b) $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx = -8$

(c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = 3(-4) = -12$

(d) $\int_2^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^2 f(x)\,dx = 6 - (-4) = 10$

(e) $\int_1^5 [f(x) - g(x)]\,dx = \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 6 - 8 = -2$

(f) $\int_1^5 [4f(x) - g(x)]\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 4(6) - 8 = 16$

10. (a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

(b) $\int_7^9 [f(x) + h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5 + 4 = 9$

(c) $\int_7^9 [2f(x) - 3h(x)]\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2(5) - 3(4) = -2$

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

(e) $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1 - 5 = -6$

(f) $\int_9^7 [h(x) - f(x)]\,dx = \int_7^9 [f(x) - h(x)]\,dx = \int_7^9 f(x)\,dx - \int_7^9 h(x)\,dx = 5 - 4 = 1$

11. (a) $\int_1^2 f(u)\,du = \int_1^2 f(x)\,dx = 5$

(b) $\int_1^2 \sqrt{3}\,f(z)\,dz = \sqrt{3}\int_1^2 f(z)\,dz = 5\sqrt{3}$

(c) $\int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -5$

(d) $\int_1^2 [-f(x)]\,dx = -\int_1^2 f(x)\,dx = -5$

12. (a) $\int_0^{-3} g(t)\,dt = -\int_{-3}^{0} g(t)\,dt = -\sqrt{2}$ (b) $\int_{-3}^{0} g(u)\,du = \int_{-3}^{0} g(t)\,dt = \sqrt{2}$

(c) $\int_{-3}^{0} [-g(x)]\,dx = -\int_{-3}^{0} g(x)\,dx = -\sqrt{2}$ (d) $\int_{-3}^{0} \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^{0} g(t)\,dt = \left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{2}\right) = 1$

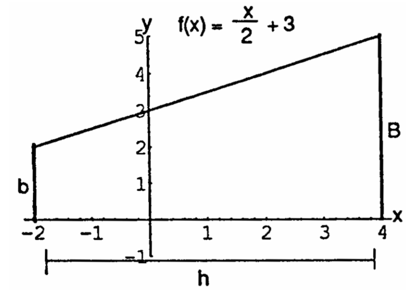
13. (a) $\int_3^4 f(z)\,dz = \int_0^4 f(z)\,dz - \int_0^3 f(z)\,dz = 7 - 3 = 4$

(b) $\int_4^3 f(t)\,dt = -\int_3^4 f(t)\,dt = -4$

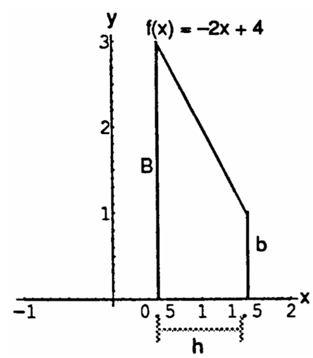
14. (a) $\int_1^3 h(r)\,dr = \int_{-1}^{3} h(r)\,dr - \int_{-1}^{1} h(r)\,dr = 6 - 0 = 6$

(b) $-\int_3^1 h(u)\,du = -\left(-\int_1^3 h(u)\,du\right) = \int_1^3 h(u)\,du = 6$

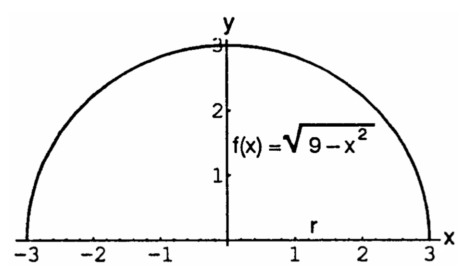
15. The area of the trapezoid is $A = \frac{1}{2}(B + b)h$
$= \frac{1}{2}(5 + 2)(6) = 21 \Rightarrow \int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$
$= 21$ square units



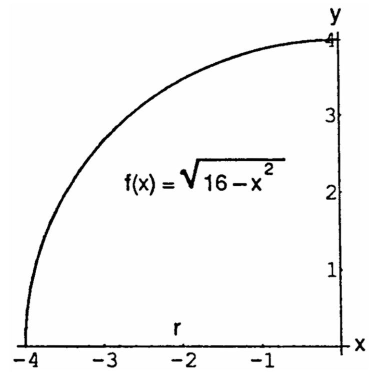
16. The area of the trapezoid is $A = \frac{1}{2}(B + b)h$
$= \frac{1}{2}(3 + 1)(1) = 2 \Rightarrow \int_{1/2}^{3/2} (-2x + 4)\,dx$
$= 2$ square units



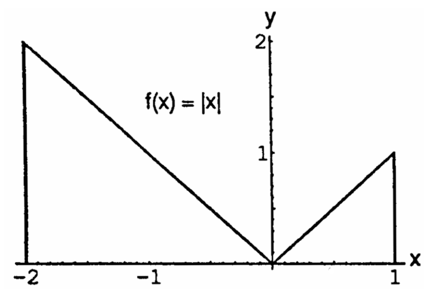
17. The area of the semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$
$= \frac{9}{2}\pi \Rightarrow \int_{-3}^{3} \sqrt{9 - x^2}\,dx = \frac{9}{2}\pi$ square units



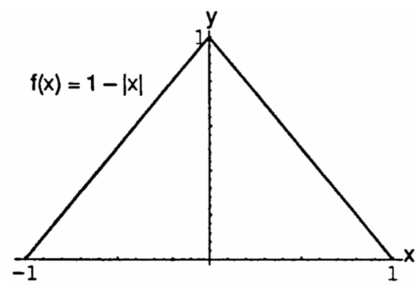
18. The graph of the quarter circle is $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(4)^2 = 4\pi \Rightarrow \int_{-4}^{0} \sqrt{16 - x^2}\, dx = 4\pi$ square units



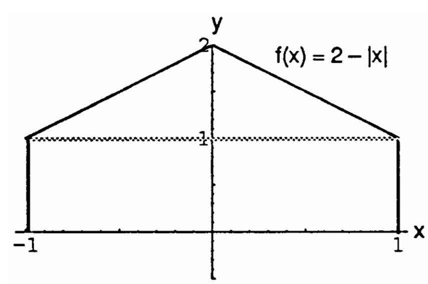
19. The area of the triangle on the left is $A = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$. The area of the triangle on the right is $A = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$. Then, the total area is 2.5 $\Rightarrow \int_{-2}^{1} |x|\, dx = 2.5$ square units



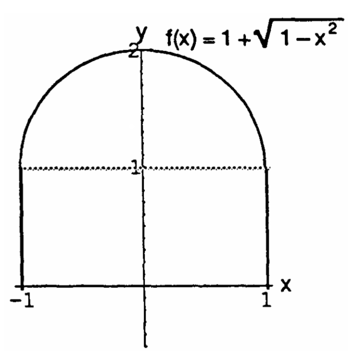
20. The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1 \Rightarrow \int_{-1}^{1} (1 - |x|)\, dx = 1$ square unit



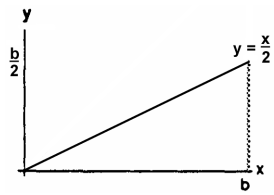
21. The area of the triangular peak is $A = \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$. The area of the rectangular base is $S = \ell w = (2)(1) = 2$. Then the total area is 3 $\Rightarrow \int_{-1}^{1} (2 - |x|)\, dx = 3$ square units



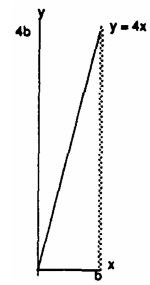
22. $y = 1 + \sqrt{1 - x^2} \Rightarrow y - 1 = \sqrt{1 - x^2} \Rightarrow (y - 1)^2 = 1 - x^2 \Rightarrow x^2 + (y - 1)^2 = 1$, a circle with center $(0, 1)$ and radius of 1 $\Rightarrow y = 1 + \sqrt{1 - x^2}$ is the upper semicircle. The area of this semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$. The area of the rectangular base is $A = \ell w = (2)(1) = 2$. Then the total area is $2 + \frac{\pi}{2}$ $\Rightarrow \int_{-1}^{1} \left(1 + \sqrt{1 - x^2}\right) dx = 2 + \frac{\pi}{2}$ square units



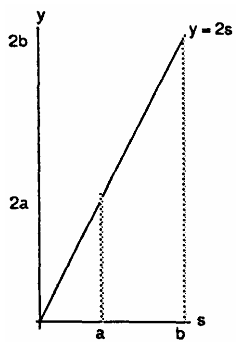
23. $\int_0^b \frac{x}{2}\,dx = \frac{1}{2}(b)\left(\frac{b}{2}\right) = \frac{b^2}{4}$



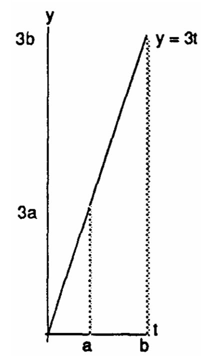
24. $\int_0^b 4x\,dx = \frac{1}{2}\,b(4b) = 2b^2$



25. $\int_a^b 2s\,ds = \frac{1}{2}\,b(2b) - \frac{1}{2}\,a(2a) = b^2 - a^2$



26. $\int_a^b 3t\,dt = \frac{1}{2}\,b(3b) - \frac{1}{2}\,a(3a) = \frac{3}{2}\,(b^2 - a^2)$



27. (a) $\int_{-2}^{2} \sqrt{4 - x^2}\,dx = \frac{1}{2}\left[\pi(2)^2\right] = 2\pi$ (b) $\int_0^2 \sqrt{4 - x^2}\,dx = \frac{1}{4}\left[\pi(2)^2\right] = \pi$

28. (a) $\int_{-1}^{0} \left(3x + \sqrt{1 - x^2}\right) dx = \int_{-1}^{0} 3x\,dx + \int_{-1}^{0} \sqrt{1 - x^2}\,dx = -\frac{1}{2}\left[(1)(3)\right] + \frac{1}{4}\left[\pi(1)^2\right] = \frac{\pi}{4} - \frac{3}{2}$

(b) $\int_{-1}^{0} \left(3x + \sqrt{1 - x^2}\right) dx = \int_{-1}^{0} 3x\,dx + \int_0^1 3x\,dx + \int_{-1}^{1} \sqrt{1 - x^2}\,dx = -\frac{1}{2}\left[(1)(3)\right] + \frac{1}{2}\left[(1)(3)\right] + \frac{1}{2}\left[\pi(1)^2\right] = \frac{\pi}{2}$

29. $\int_1^{\sqrt{2}} x\,dx = \frac{\left(\sqrt{2}\right)^2}{2} - \frac{(1)^2}{2} = \frac{1}{2}$

30. $\int_{0.5}^{2.5} x\,dx = \frac{(2.5)^2}{2} - \frac{(0.5)^2}{2} = 3$

31. $\int_{\pi}^{2\pi} \theta\,d\theta = \frac{(2\pi)^2}{2} - \frac{\pi^2}{2} = \frac{3\pi^2}{2}$

32. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \frac{\left(5\sqrt{2}\right)^2}{2} - \frac{\left(\sqrt{2}\right)^2}{2} = 24$

33. $\int_0^{\sqrt[3]{7}} x^2\,dx = \frac{\left(\sqrt[3]{7}\right)^3}{3} = \frac{7}{3}$

34. $\int_0^{0.3} s^2\,ds = \frac{(0.3)^3}{3} = 0.009$

35. $\int_0^{1/2} t^2\,dt = \frac{\left(\frac{1}{2}\right)^3}{3} = \frac{1}{24}$

36. $\int_0^{\pi/2} \theta^2\,d\theta = \frac{\left(\frac{\pi}{2}\right)^3}{3} = \frac{\pi^3}{24}$

37. $\int_a^{2a} x\,dx = \frac{(2a)^2}{2} - \frac{a^2}{2} = \frac{3a^2}{2}$

38. $\int_a^{\sqrt{3}a} x\,dx = \frac{\left(\sqrt{3}a\right)^2}{2} - \frac{a^2}{2} = a^2$

39. $\int_0^{\sqrt[3]{b}} x^2\,dx = \frac{\left(\sqrt[3]{b}\right)^3}{3} = \frac{b}{3}$

40. $\int_0^{3b} x^2\,dx = \frac{(3b)^3}{3} = 9b^3$

41. $\int_3^1 7\,dx = 7(1-3) = -14$

42. $\int_0^2 5x\,dx = 5\int_0^2 x\,dx = 5\left[\frac{2^2}{2} - \frac{0^2}{2}\right] = 10$

43. $\int_0^2 (2t-3)\,dt = 2\int_1^1 t\,dt - \int_0^2 3\,dt = 2\left[\frac{2^2}{2} - \frac{0^2}{2}\right] - 3(2-0) = 4 - 6 = -2$

44. $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right) dt = \int_0^{\sqrt{2}} t\,dt - \int_0^{\sqrt{2}} \sqrt{2}\,dt = \left[\frac{\left(\sqrt{2}\right)^2}{2} - \frac{0^2}{2}\right] - \sqrt{2}\left[\sqrt{2} - 0\right] = 1 - 2 = -1$

45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz = \int_2^1 1\,dz + \int_2^1 \frac{z}{2}\,dz = \int_2^1 1\,dz - \frac{1}{2}\int_1^2 z\,dz = 1[1-2] - \frac{1}{2}\left[\frac{2^2}{2} - \frac{1^2}{2}\right] = -1 - \frac{1}{2}\left(\frac{3}{2}\right) = -\frac{7}{4}$

46. $\int_3^0 (2z-3)\,dz = \int_3^0 2z\,dz - \int_3^0 3\,dz = -2\int_0^3 z\,dz - \int_3^0 3\,dz = -2\left[\frac{3^2}{2} - \frac{0^2}{2}\right] - 3[0-3] = -9 + 9 = 0$

47. $\int_1^2 3u^2\,du = 3\int_1^2 u^2\,du = 3\left[\int_0^2 u^2\,du - \int_0^1 u^2\,du\right] = 3\left(\left[\frac{2^3}{3} - \frac{0^3}{3}\right] - \left[\frac{1^3}{3} - \frac{0^3}{3}\right]\right) = 3\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = 3\left(\frac{7}{3}\right) = 7$

48. $\int_{1/2}^1 24u^2\,du = 24\int_{1/2}^1 u^2\,du = 24\left[\int_0^1 u^2\,du - \int_0^{1/2} u^2\,du\right] = 24\left[\frac{1^3}{3} - \frac{\left(\frac{1}{2}\right)^3}{3}\right] = 24\left[\frac{\left(\frac{7}{8}\right)}{3}\right] = 7$

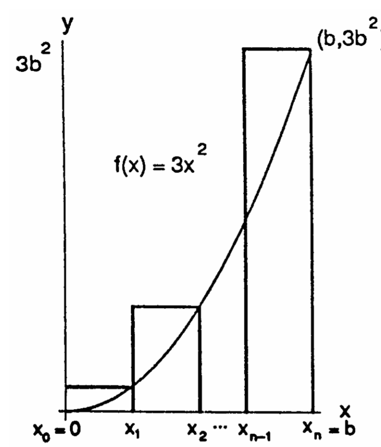
49. $\int_0^2 (3x^2 + x - 5)\,dx = 3\int_0^2 x^2\,dx + \int_0^2 x\,dx - \int_0^2 5\,dx = 3\left[\frac{2^3}{3} - \frac{0^3}{3}\right] + \left[\frac{2^2}{2} - \frac{0^2}{2}\right] - 5[2-0] = (8+2) - 10 = 0$

50. $\int_1^0 (3x^2 + x - 5)\,dx = -\int_0^1 (3x^2 + x - 5)\,dx = -\left[3\int_0^1 x^2\,dx + \int_0^1 x\,dx - \int_0^1 5\,dx\right]$
$= -\left[3\left(\frac{1^3}{3} - \frac{0^3}{3}\right) + \left(\frac{1^2}{2} - \frac{0^2}{2}\right) - 5(1-0)\right] = -\left(\frac{3}{2} - 5\right) = \frac{7}{2}$

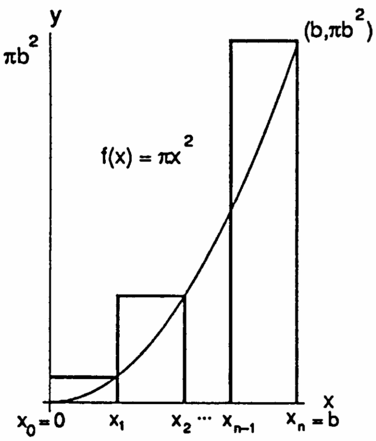
51. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_{n-1} = (n-1)\Delta x$, $x_n = n\Delta x = b$. Let the c_k's be the right end-points of the subintervals $\Rightarrow c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:

$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = 3(\Delta x)^2\,\Delta x = 3(\Delta x)^3$
$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = 3(2\Delta x)^2\,\Delta x = 3(2)^2(\Delta x)^3$
$f(c_3)\,\Delta x = f(3\Delta x)\,\Delta x = 3(3\Delta x)^2\,\Delta x = 3(3)^2(\Delta x)^3$
$\vdots$
$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = 3(n\Delta x)^2\,\Delta x = 3(n)^2(\Delta x)^3$

Then $S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} 3k^2(\Delta x)^3$
$= 3(\Delta x)^3 \sum_{k=1}^{n} k^2 = 3\left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)$
$= \frac{b^3}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \Rightarrow \int_0^b 3x^2\,dx = \lim_{n\to\infty} \frac{b^3}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = b^3.$



52. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_{n-1} = (n-1)\Delta x$, $x_n = n\Delta x = b$. Let the c_k's be the right end-points of the subintervals $\Rightarrow$ $c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:



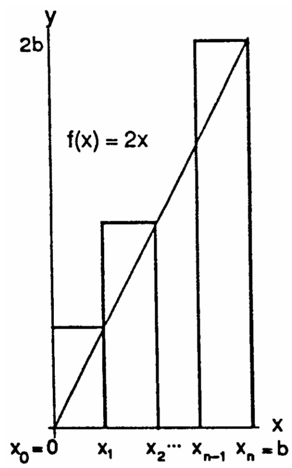
$$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = \pi(\Delta x)^2\,\Delta x = \pi(\Delta x)^3$$
$$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = \pi(2\Delta x)^2\,\Delta x = \pi(2)^2(\Delta x)^3$$
$$f(c_3)\,\Delta x = f(3\Delta x)\,\Delta x = \pi(3\Delta x)^2\,\Delta x = \pi(3)^2(\Delta x)^3$$
$$\vdots$$
$$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = \pi(n\Delta x)^2\,\Delta x = \pi(n)^2(\Delta x)^3$$

Then $S_n = \sum\limits_{k=1}^{n} f(c_k)\,\Delta x = \sum\limits_{k=1}^{n} \pi k^2(\Delta x)^3$

$= \pi(\Delta x)^3 \sum\limits_{k=1}^{n} k^2 = \pi\left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{\pi b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \Rightarrow \int_0^b \pi x^2\,dx = \lim\limits_{n\to\infty} \frac{\pi b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{\pi b^3}{3}.$

53. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_{n-1} = (n-1)\Delta x$, $x_n = n\Delta x = b$. Let the c_k's be the right end-points of the subintervals $\Rightarrow$ $c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:



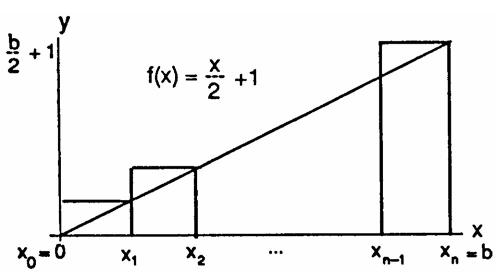
$$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = 2(\Delta x)(\Delta x) = 2(\Delta x)^2$$
$$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = 2(2\Delta x)(\Delta x) = 2(2)(\Delta x)^2$$
$$f(c_3)\,\Delta x = f(3\Delta x)\,\Delta x = 2(3\Delta x)(\Delta x) = 2(3)(\Delta x)^2$$
$$\vdots$$
$$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = 2(n\Delta x)(\Delta x) = 2(n)(\Delta x)^2$$

Then $S_n = \sum\limits_{k=1}^{n} f(c_k)\,\Delta x = \sum\limits_{k=1}^{n} 2k(\Delta x)^2$

$= 2(\Delta x)^2 \sum\limits_{k=1}^{n} k = 2\left(\frac{b^2}{n^2}\right)\left(\frac{n(n+1)}{2}\right)$

$= b^2\left(1 + \frac{1}{n}\right) \Rightarrow \int_0^b 2x\,dx = \lim\limits_{n\to\infty} b^2\left(1 + \frac{1}{n}\right) = b^2.$

54. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_{n-1} = (n-1)\Delta x$, $x_n = n\Delta x = b$. Let the c_k's be the right end-points of the subintervals $\Rightarrow$ $c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:

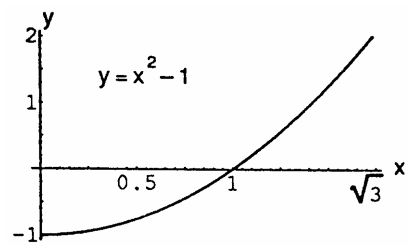


$$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = \left(\frac{\Delta x}{2} + 1\right)(\Delta x) = \frac{1}{2}(\Delta x)^2 + \Delta x$$
$$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = \left(\frac{2\Delta x}{2} + 1\right)(\Delta x) = \frac{1}{2}(2)(\Delta x)^2 + \Delta x$$
$$f(c_3)\,\Delta x = f(3\Delta x)\,\Delta x = \left(\frac{3\Delta x}{2} + 1\right)(\Delta x) = \frac{1}{2}(3)(\Delta x)^2 + \Delta x$$
$$\vdots$$
$$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = \left(\frac{n\Delta x}{2} + 1\right)(\Delta x) = \frac{1}{2}(n)(\Delta x)^2 + \Delta x$$

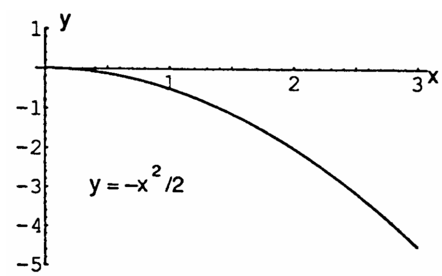
Then $S_n = \sum\limits_{k=1}^{n} f(c_k)\,\Delta x = \sum\limits_{k=1}^{n}\left(\frac{1}{2}k(\Delta x)^2 + \Delta x\right) = \frac{1}{2}(\Delta x)^2 \sum\limits_{k=1}^{n} k + \Delta x \sum\limits_{k=1}^{n} 1 = \frac{1}{2}\left(\frac{b^2}{n^2}\right)\left(\frac{n(n+1)}{2}\right) + \left(\frac{b}{n}\right)(n)$

$= \frac{1}{4}b^2\left(1 + \frac{1}{n}\right) + b \Rightarrow \int_0^b \left(\frac{x}{2} + 1\right)dx = \lim\limits_{n\to\infty}\left(\frac{1}{4}b^2\left(1 + \frac{1}{n}\right) + b\right) = \frac{1}{4}b^2 + b.$

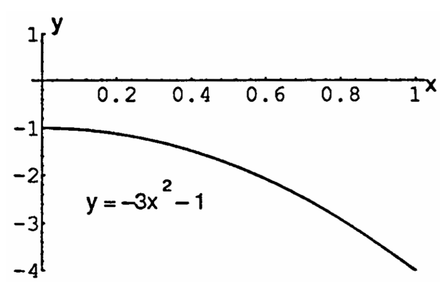
55. $\text{av}(f) = \left(\frac{1}{\sqrt{3}-0}\right)\int_0^{\sqrt{3}} (x^2-1)\,dx$
$= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} x^2\,dx - \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} 1\,dx$
$= \frac{1}{\sqrt{3}}\left(\frac{\left(\sqrt{3}\right)^3}{3}\right) - \frac{1}{\sqrt{3}}\left(\sqrt{3}-0\right) = 1 - 1 = 0.$



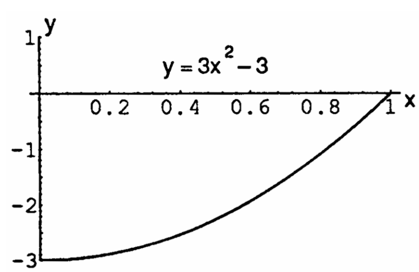
56. $\text{av}(f) = \left(\frac{1}{3-0}\right)\int_0^3 \left(-\frac{x^2}{2}\right)dx = \frac{1}{3}\left(-\frac{1}{2}\right)\int_0^3 x^2\,dx$
$= -\frac{1}{6}\left(\frac{3^3}{3}\right) = -\frac{3}{2}\,;\ -\frac{x^2}{2} = -\frac{3}{2}.$



57. $\text{av}(f) = \left(\frac{1}{1-0}\right)\int_0^1 (-3x^2-1)\,dx =$
$= -3\int_0^1 x^2\,dx - \int_0^1 1\,dx = -3\left(\frac{1^3}{3}\right) - (1-0)$
$= -2.$



58. $\text{av}(f) = \left(\frac{1}{1-0}\right)\int_0^1 (3x^2-3)\,dx =$
$= 3\int_0^1 x^2\,dx - \int_0^1 3\,dx = 3\left(\frac{1^3}{3}\right) - 3(1-0)$
$= -2.$



59. $\text{av}(f) = \left(\frac{1}{3-0}\right)\int_0^3 (t-1)^2\,dt$
$= \frac{1}{3}\int_0^3 t^2\,dt - \frac{2}{3}\int_0^3 t\,dt + \frac{1}{3}\int_0^3 1\,dt$
$= \frac{1}{3}\left(\frac{3^3}{3}\right) - \frac{2}{3}\left(\frac{3^2}{2} - \frac{0^2}{2}\right) + \frac{1}{3}(3-0) = 1.$